

On approximation of Ginzburg-Landau minimizers by \mathbb{S}^1 -valued maps in domains with vanishingly small holes

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Abstract: We consider a two-dimensional Ginzburg-Landau problem on an arbitrary domain with a finite number of vanishingly small circular holes. A special choice of scaling relation between the material and geometric parameters (Ginzburg-Landau parameter vs hole radius) is motivated by a recently discovered phenomenon of vortex phase separation in superconducting composites. We show that, for each hole, the degrees of minimizers of the Ginzburg-Landau problems in the classes of \mathbb{S}^1 -valued and \mathbb{C} -valued maps, respectively, are the same. The presence of two parameters that are widely separated on a logarithmic scale constitutes the principal difficulty of the analysis that is based on energy decomposition techniques.

1 Introduction

The present study is motivated by the pinning phenomenon in type-II superconducting composites. Type-II superconductors are characterized by vanishing resistivity and complete expulsion of magnetic fields from the bulk of the material at sufficiently low temperatures. When the magnitude h_{ext} of an external magnetic field \mathbf{h}_{ext} exceeds a certain threshold, the field begins to penetrate the superconductor along isolated vortex lines that may move, resulting in energy dissipation. This motion and related energy losses can be inhibited by pinning the lines to impurities or holes in a *superconducting composite*. Understanding the role of imperfections in a superconductor can thus be used to design more efficient superconducting materials. In what follows, we will consider a cylindrical superconducting sample containing rod-like inclusions or *columnar defects* elongated along the axis of the cylinder, so that the sample can be represented by its cross-section $\Omega \subset \mathbb{R}^2$. Then the vortex lines penetrate each cross-section at isolated points, called *vortices*.

Superconductivity is typically modeled within the framework of the Ginzburg-Landau theory [11] in terms of an order-parameter $u \in \mathbb{C}$ and the vector potential of the induced magnetic field $A \in \mathbb{R}^2$. The appearance and behavior of vortices for the minimizers of the Ginzburg-Landau functional

$$GL^\varepsilon[u, A] = \frac{1}{2} \int_{\Omega} |(\nabla - iA)u|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 dx + \frac{1}{2} \int_{\Omega} (\operatorname{curl} A - h_{ext})^2 dx \quad (1)$$

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have been studied, in particular, in [16, 18] where the existence of two critical magnetic fields, H_{c1} and H_{c2} , was established rigorously for simply-connected domain when $\varepsilon > 0$ is small. When the external magnetic field is weak ($h_{ext} < H_{c1}$) it is completely expelled from the bulk semiconductor (Meissner effect) and there are no vortices. When the field strength is ramped up from H_{c1} to H_{c2} , the magnetic field penetrates the superconductor through an increasing number of isolated vortices while the superconductivity is destroyed everywhere, once the field exceeds H_{c2} .

The pinning phenomenon that we consider in this paper is observed in non-simply-connected domains with holes that may or may not contain another material. If a hole "pins" a vortex, the order parameter u has a nonzero winding number on the boundary of the hole. We refer to this object as a *hole vortex*. Note that degrees of the hole vortices increase along with the strength of the external magnetic field. This situation is in contrast with the regular bulk vortices that have degree ± 1 and increase in number as the field becomes stronger.

An alternative way to model the impurities is to consider a potential term $(a(x) - |u|^2)^2$ where $a(x)$ varies throughout the sample. It was proven in [9] that the impurities corresponding to the weakest superconductivity (where $a(x)$ is minimal) pin the vortices first. This model was studied further in [1] and [4] to demonstrate the existence of nontrivial pinning patterns and in [2] to investigate the breakdown of pinning in an increasing external magnetic field, among other issues. A composite consisting of two superconducting samples with different critical temperatures was considered in [5, 14] where nucleation of vortices near the interface was shown to occur.

In our model we consider a superconductor with holes, similar to the setup in [3]. In that work, the authors considered the asymptotic limits of minimizers of GL^ε as $\varepsilon \rightarrow 0$ and determined that holes act as pinning sites gaining nonzero degree for moderate but bounded magnetic fields. For magnetic fields below the threshold of order $|\ln \varepsilon|$ the degree of the order parameter on the holes continues to grow without bound, however beyond the critical field strength, the pinning breaks down and vortices appear in the interior of the superconductor. Since the contribution to the energy from the hole vortices has a logarithmic dependence on the diameter of the holes, the hole size can be used as an additional small parameter to enforce a finite degree of the hole vortex in the limit of small ε . The domain with finitely many shrinking (pinning) subdomains with weakened superconductivity was considered in [10] in the case of the simplified Ginzburg-Landau functional. The model with a potential term $(a(x) - |u|^2)^2$ with piecewise constant $a(x)$ was used to enforce pinning and it was observed that the vortices are localized within pinning domains and converge to their centers.

The problem considered in this work was inspired by the result in [6] where a periodic lattice of vanishingly small holes was considered. The main interest was in the regime when the radii of the holes were exponentially small compared to the period a of the lattice; both of these parameters were assumed to converge to zero along with ε . Using homogenization-type arguments, it was shown in [6] that in the limit of $\varepsilon \rightarrow 0$ and when the external magnetic field of order $O(a^{-2})$, the minimizers can be characterized by nested subdomains of constant vorticity. The physical nature of this result was discussed in [12]. The analysis in [6] relies on a conjecture that for small ε , the degrees of the hole vortices are the same for both \mathbb{C} - and \mathbb{S}^1 -valued maps. The principal aim of the present paper is to establish the validity of this conjecture in the case of finitely many vanishingly small holes.

Our approach builds on that of [3], combined with the appropriately chosen lower bounds on the energy and the ball construction method [7], [13], [15]-[19].

The paper is organized as follows. Section 2 contains the formulation of the problem, as well as the main result described in Theorem 1. In Section 3, we prove that the minimizers in the class

of \mathbb{S}^1 -valued maps are characterized by the unique set of integer degrees on the holes. In Section 4, we use the approach, similar to that in [3], to express the energy of a \mathbb{C} -valued minimizer as the sum of the energy of the S^1 -valued minimizer and the remainder terms. Compared to [3], additional complications arise in the analysis due to the fact that the radius of the holes is not fixed in the present work. In particular, because of the presence of another small parameter, we use a different ball construction method that incorporates both the Ginzburg-Landau parameter ε and the holes radius δ . In Section 5 we show that the minimizers cannot have vortices with nonzero degrees outside of the holes. This section also provides sharp energy estimates that allow us to prove the main theorem. Finally, in Section 6, the equality of degrees is established based on the estimates obtained in the previous section.

2 Main results

Let $B(x_0, R) \subset \mathbb{R}^2$ denote a disk of radius R centered at x_0 . Let Ω be an arbitrary smooth, bounded, simply connected domain and suppose that $\omega_\delta^j = B(a^j, \delta) \subset \Omega$, $j = 1 \dots N$ represent the holes in Ω , where a^j is the center of the hole $j = 1, \dots, N$ and $\delta \ll 1$ is its radius. We introduce the perforated domain

$$\Omega_\delta = \Omega \setminus \bigcup_{j=1}^N \omega_\delta^j \quad (2)$$

and consider the Ginzburg-Landau functional

$$GL_\delta^\varepsilon[u, A] = \frac{1}{2} \int_{\Omega_\delta} |(\nabla - iA)u|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega_\delta} (1 - |u|^2)^2 dx + \frac{1}{2} \int_{\Omega} (\text{curl } A - h_{ext})^2 dx. \quad (3)$$

The domain Ω_δ represents a cross-section of a superconducting sample. Here $u : \Omega_\delta \rightarrow \mathbb{C}$ is an order parameter, $A : \Omega \rightarrow \mathbb{R}^2$ is a vector potential of the induced magnetic field, and h_{ext} is the magnitude of the external magnetic field. By ε we denote the inverse of the Ginzburg-Landau parameter that determines the radius of a typical vortex core. In what follows, we will assume that the cores radii are much smaller than the radius of the holes ω_δ^j .

The functional $GL_\delta^\varepsilon[u, A]$ is gauge-invariant, i.e., for any $\varphi \in H^2(\Omega, \mathbb{R})$ and any admissible pair (u, A) , the equality $GL_\delta^\varepsilon[u, A] = GL_\delta^\varepsilon[u e^{i\varphi}, A + \nabla\varphi]$ always holds. This degeneracy can be eliminated by imposing the *Coulomb gauge*, that is requiring that

$$A \in H(\Omega, \mathbb{R}^2) := \{a \in H^1(\Omega, \mathbb{R}^2) \mid \text{div } a = 0 \text{ in } \Omega, \ a \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad (4)$$

where ν is an outward unit normal vector to $\partial\Omega$. We will fix the Coulomb gauge throughout the rest of this work.

We consider the minimizers of the two variational problems

$$(u_\delta^\varepsilon, A_\delta^\varepsilon) := \arg \min \{GL_\delta^\varepsilon[u, A] \mid u \in H^1(\Omega_\delta; \mathbb{C}), A \in H(\Omega; \mathbb{R}^2)\}, \quad (5)$$

and

$$(u_\delta, A_\delta) := \arg \min \{GL_\delta^\varepsilon[u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2)\}. \quad (6)$$

Note that, trivially,

$$(u_\delta, A_\delta) := \arg \min \{GL_\delta[u, A] \mid u \in H^1(\Omega_\delta; S^1), A \in H(\Omega; \mathbb{R}^2)\}, \quad (7)$$

where

$$GL_\delta[u, A] = \frac{1}{2} \int_{\Omega_\delta} |\nabla u - iAu|^2 dx + \frac{1}{2} \int_{\Omega} (\operatorname{curl} A - h_{ext})^2 dx. \quad (8)$$

For any hole center a^j , $j = 1, \dots, N$ and $R > 0$, let $\gamma_R^j = \partial B(a^j, R)$ be a circle of radius R centered at a^j . In what follows we make a frequent use of the following

Definition 1. Given a $u \in H^1(\Omega_\delta, \mathbb{C})$ and a^j , $j = 1, \dots, N$, suppose there exists an $R = \delta + o(\delta)$ such that the winding number $d = \deg(u/|u|, \gamma_R^j) \neq 0$. Then u is said to have a *hole vortex* of the degree d inside ω_δ^j .

The existence of γ_R^j is established in the Theorem 1 and they are specified using the results of Theorem 3. Hole vortices may exist inside ω_δ^j for the minimizers of both (5) and (7) and our principal goal is to prove that the respective degrees of the hole vortices arising in both problems coincide for the same external magnetic field as long as the parameter δ is sufficiently small. This result implies that the non-linear potential term can be effectively replaced by the constraint $|u| = 1$ when one is interested in studying the distribution of degrees of the hole vortices for the minimizer of the problem (5).

The main result of this work is the following theorem.

Theorem 1. Assume that the parameters ε and δ satisfy

$$|\log \varepsilon| \gg |\log \delta|. \quad (9)$$

Suppose

$$\sigma \in \mathbb{R}_+ \setminus \Sigma \quad (10)$$

where Σ is a discrete set described below. Let

$$h_{ext} = \sigma |\log \delta| \quad (11)$$

and $(u_\delta^\varepsilon, A_\delta^\varepsilon)$ and (u_δ, A_δ) be defined by (5) and (7), respectively.

Then, for a sufficiently small δ , there exists an $R_\delta \in [\delta, \delta + \delta^2]$ such that

- (i) $D_\delta^j = \deg(u_\delta, \gamma_R^j)$ coincide for all $j = 1 \dots N$ when $D_{\delta, \varepsilon}^j$ are defined, e.g. when $u_\delta^\varepsilon \neq 0$ on γ_R^j ;
- (ii) the degrees of the hole vortices $D_{\delta, \varepsilon}^j = \deg\left(\frac{u_\delta^\varepsilon}{|u_\delta^\varepsilon|}, \gamma_R^j\right)$;

for any $R \geq R_\delta$ for which $\gamma_R^j = \partial B(a^j, R)$, $j = 1 \dots N$ are mutually disjoint and do not intersect $\partial\Omega$.

Remark 1. The set Σ includes the appropriately scaled values of the external field at which the degree of one of the hole vortices increments by one, i.e. from d to $d + 1$. At these threshold field strengths, the leading order approximation of the energy is the same for both degrees d and $d + 1$ and the degrees of the hole vortices of minimizers u_δ^ε and u_δ cannot be determined uniquely. The set Σ is described as follows:

$$\Sigma = \bigcup_{j=1}^N \Sigma_j \quad \text{where} \quad \Sigma_j = \left\{ \sigma > 0 \mid \sigma (1 - \xi_0(a^j)) \in \mathbb{Z} + \frac{1}{2} \right\} \quad (12)$$

consists of the threshold field values for the hole $j = 1 \dots N$ and the function ξ_0 solves the boundary value problem

$$\begin{cases} -\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega, \\ \xi_0 = 1 & \text{on } \partial\Omega. \end{cases} \quad (13)$$

Remark 2. Notice that, since $u_\delta(x) \in \mathbb{S}^1$, there are no vortices outside of the holes and thus

$$D_\delta^j = \deg(u_\delta, \gamma_r^j) = \deg(u_\delta, \partial\omega_\delta^j) \quad (14)$$

for all $j = 1 \dots N$.

Remark 3. As we will show in Section 5, although the external magnetic field satisfying the bound (11) is strong enough to generate hole vortices, it is too weak for vortices to appear inside the bulk superconductor Ω_δ , away from the boundary $\partial\Omega$.

We prove Theorem 1 in two steps. First, we consider minimizers $(u_{\delta D}, A_{\delta D})$ of the variational problem (8) in the class of \mathbb{S}^1 -valued maps with the prescribed degrees, $\deg(u, \partial\omega_\delta^j) = D^j$, $j = 1 \dots N$, by setting

$$(u_{\delta D}, A_{\delta D}) := \arg \min \left\{ GL_\delta[u, A] \mid u \in H^1(\Omega_\delta; \mathbb{S}^1), A \in H(\Omega; \mathbb{R}^2), \deg(u, \partial\omega_\delta^j) = D^j \right\}. \quad (15)$$

Then the degrees D_δ^j of the map u_δ minimize the energy

$$l_\delta(D) := GL_\delta[u_{\delta D}, A_{\delta D}] \quad (16)$$

where $D = (D^1, \dots, D^N)$. It turns out that the function $l_\delta(D)$ is a quadratic polynomial in D^1, \dots, D^N . Its minimum is attained at one of the integer points adjacent to the vertex of paraboloid $l_\delta(T)$ with $T \in \mathbb{R}^N$. We enforce the condition (10) to ensure that such minimizing integer point is unique.

We then express a minimizer $(u_\delta^\varepsilon, A_\delta^\varepsilon)$ of $GL_\delta^\varepsilon[u, A]$ as a sum of (u_δ, A_δ) and an appropriate correction term and consider a corresponding energy decomposition in the spirit of the approach in [3] for finite-size holes. The analysis relies principally on the techniques developed in [3] and the ball construction method [19]. Compared to [3], new challenges arise due to the presence of the second small parameter that require additional estimates and sharper energy bounds.

3 \mathbb{S}^1 -valued problem

The main goal of this section is to establish the relation between the energy of the minimizer $(u_{\delta D}, A_{\delta D})$ and the degrees D of the hole vortices corresponding to $u_{\delta D}$. We approximate the minimizer $(u_{\delta D}, A_{\delta D})$, calculate its energy $l_\delta(D) = GL_\delta[u_{\delta D}, A_{\delta D}]$, and find the minimizing degrees $D_\delta = (D_\delta^1, \dots, D_\delta^N)$. We prove the following theorem.

Theorem 2. *Let $(u_{\delta D}, A_{\delta D})$ be a minimizer of (15) with the prescribed degrees $D \in \mathbb{Z}^N$ on the holes. Then the Ginzburg-Landau energy $GL_\delta[u_{\delta D}, A_{\delta D}]$, expressed as a function of D , takes the following form:*

$$l_\delta(D) = \pi \sum_{j=1}^N \left[(D^j)^2 - 2\sigma(1 - \xi_0(a^j)) D^j \right] |\log \delta| + C |\log \delta|^2 + |D|^2 O(1) \quad (17)$$

where ξ_0 solves the boundary value problem (13), $C = O(1)$, and $|D| = \max_j |D^j|$.

Proof. The main idea of the proof is to approximate the induced magnetic field $h_{\delta D} = \text{curl } A_{\delta D}$ as a sum of functions that depend on external magnetic field and the prescribed degrees on the holes, respectively. First, prescribe the degrees of the order parameter

$$\deg(u, \partial\omega_\delta^j) = D^j, \quad j = 1 \dots N \quad (18)$$

and write down the Euler-Lagrange equation for (8) in terms of the induced magnetic field $h = \text{curl } A$ with the corresponding boundary conditions:

$$\begin{cases} -\Delta h + h = 0, & \text{in } \Omega_\delta, \\ h = h_{ext}, & \text{on } \partial\Omega, \\ h = H_j, & \text{in } \omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial h}{\partial \nu} ds = 2\pi D^j - \int_{\omega_\delta^j} h dx, & j = 1 \dots N. \end{cases} \quad (19)$$

The constants H_j are a priori unknown and are defined through the solution $h_{\delta D} = h_\delta(D)$ of (19) where $D = (D^1, \dots, D^N)$ is the vector of the prescribed degrees. The energy (8) of the minimizer $(u_{\delta D}, A_{\delta D})$ can be expressed in terms of $h_{\delta D}$:

$$GL_\delta[u_{\delta D}, A_{\delta D}] = GL_\delta[h_{\delta D}] = \frac{1}{2} \int_{\Omega_\delta} |\nabla h_{\delta D}|^2 dx + \frac{1}{2} \int_{\Omega} (h_{\delta D} - h_{ext})^2 dx. \quad (20)$$

Decompose the solution of (19) $h_{\delta D}$ into

$$h_{\delta D} = h_1 + h_2 + h_3, \quad (21)$$

where h_1 captures the influence of the external field h_{ext} , h_2 takes into account the hole vortices, and h_3 is the remainder. More precisely,

$$h_1 = h_{ext} \xi_0, \quad (22)$$

where ξ_0 solves the boundary value problem (13) in the domain Ω with no holes:

$$\begin{cases} -\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega, \\ \xi_0 = 1 & \text{on } \partial\Omega. \end{cases} \quad (23)$$

The function h_2 is defined by

$$h_2(x) = \sum_{j=1}^N D^j \theta_j(x) \phi_j(x) \quad (24)$$

where D^j are as in (18). Here

$$\theta_j(x) = \theta(x - a^j), \quad j = 1, \dots, N$$

and θ is a truncated modified Bessel function of the second kind

$$\theta(x) = \begin{cases} K_0(\delta), & |x| \leq \delta, \\ K_0(|x|), & |x| > \delta. \end{cases} \quad (25)$$

The cutoff function $\phi_j(x) = \phi(x - a^j) \in C^\infty(\mathbb{R}^2)$ satisfies

$$\phi(x) = \begin{cases} 1, & |x| \leq R/4, \\ 0, & |x| \geq R/2, \end{cases} \quad (26)$$

with R being defined as the largest radius for which $B(a^j, R)$, $j = 1 \dots N$ intersect neither each other nor the boundary $\partial\Omega$. Here the choice of $K_0(|x|)$ is motivated by the fact that it is a fundamental solution of the equation $-\Delta u + u = 2\pi\delta(x)$ in \mathbb{R}^2 . Note that h_2 solves the following problem:

$$\begin{cases} -\Delta h_2 + h_2 = \sum_{j=1}^N D^j [-\Delta + I] (\theta_j \phi_j), & \text{in } \Omega_\delta, \\ h_2 = 0, & \text{on } \partial\Omega, \\ h_2 = D^j K_0(\delta), & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial h_2}{\partial \nu} ds = 2\pi D^j - D^j K_0(\delta) |\omega_\delta^j| + D^j O(\delta^2), & j = 1 \dots N. \end{cases} \quad (27)$$

Since for each $j = 1, \dots, N$ the function $f_j(x) := [-\Delta + I] (\theta_j \phi_j)$ is nonzero only inside the annular region $T_j := B(a^j, R/2) \setminus \overline{B(a^j, R/4)}$ that does not intersect any of the holes, the functions f_j , $j = 1, \dots, N$ are smooth and finite. Thus, for every $j = 1, \dots, N$, the function h_2 has the degree D^j on the hole ω_δ^j and $\theta_j \phi_j$ is constant on ω_δ^j and decays to zero on $\partial B(a^j, R/2)$.

Next, we show that the contribution of the remainder $h_3 = h - h_1 - h_2$ to the energy is small, hence the interaction between the hole vortices contributes a negligible amount to the energy. This provides a justification for treating each hole vortex as being independent from the other hole vortices.

We deduce the boundary value problem for h_3 from the original problem (19), the problem (13) for $h_1 = h_{ext}\xi_0$, and the expression (24) for h_2 to obtain:

$$\begin{cases} -\Delta h_3 + h_3 = -\sum_{j=1}^N D^j f_j(x), & \text{in } \Omega_\delta, \\ h_3 = 0, & \text{on } \partial\Omega, \\ h_3 = \tilde{H}_j - h_{ext}(\xi_0(x) - \xi_0(a^j)), & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial h_3}{\partial \nu} ds = -\tilde{H}_j |\omega_\delta^j| + D^j O(\delta^2) + O(\delta^3 \log \delta), & j = 1 \dots N. \end{cases} \quad (28)$$

where $\tilde{H}_j = H_j - h_{ext}\xi_0(a^j) - D^j K_0(\delta)$ are the unknown constants. The next lemma establishes the necessary estimates for h_3 .

Lemma 1. *The solution h_3 of (28) satisfies the following estimates:*

$$\|h_3\|_{L^\infty(\Omega)} \leq C_1 \delta |\log \delta|^2 + C_2 |D|, \quad (29)$$

$$\|\nabla h_3\|_{L^\infty(\Omega)} \leq C_1 |\log \delta|^2 + C_2 |D| |\log \delta|, \quad (30)$$

$$\left| \frac{\partial h_3}{\partial \nu} \right| \leq C_1 |\log \delta| + C_2 |D| \text{ on } \partial\omega_\delta^j \text{ for all } j = 1 \dots N. \quad (31)$$

Proof. We begin by splitting (28) into several subproblems. First, let $\eta = \sum_{j=1}^N D^j \eta_j$ be a solution of the nonhomogeneous equation in (28), where η_j solves

$$\begin{cases} -\Delta \eta_j + \eta_j = -[-\Delta + I] (\theta_j \phi_j) \mathbb{1}_{T_j}, & \text{in } \Omega, \\ \eta_j = 0, & \text{on } \partial\Omega, \end{cases} \quad (32)$$

for every $j = 1, \dots, N$. Here η_j , $j = 1, \dots, N$ are smooth and do not depend on δ . Next, introduce η_0 that both solves the homogeneous equation and satisfies the conditions on $\partial\omega_\delta^j$ in (28) to give

$$\begin{cases} -\Delta\eta_0 + \eta_0 = 0, & \text{in } \Omega_\delta, \\ \eta_0 = 0, & \text{on } \partial\Omega, \\ \eta_0 = -h_{ext}(\xi_0(x) - \xi_0(a^j)) - (\eta(x) - \eta(a^j)), & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N. \end{cases} \quad (33)$$

Note that, by the Maximum Principle,

$$\|\eta_0\|_{L^\infty} \leq C\delta(|\log \delta| + \max_j |D^j|). \quad (34)$$

Lemma 6 provides the estimate on the gradient of η_0 of the form

$$\|\nabla\eta_0\|_{L^\infty} \leq C(|\log \delta| + \max_j |D^j|). \quad (35)$$

The remainder $\zeta = h_3 - \sum_{j=0}^N \eta_j$ solves the following system:

$$\begin{cases} -\Delta\zeta + \zeta = 0, & \text{in } \Omega_\delta, \\ \zeta = 0, & \text{on } \partial\Omega, \\ \zeta = c_j, & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \\ -\int_{\partial\omega_\delta^j} \frac{\partial\zeta}{\partial\nu} ds = -|\omega_\delta^j|c_j + A_\delta^j, & j = 1 \dots N, \end{cases} \quad (36)$$

where $c_j = \tilde{H}_j - \eta(a^j)$ are unknown constants and $A_\delta^j = |D|O(\delta) + O(\delta \log \delta)$ is an error. The first three equations in (36) set up the boundary value problem for ζ with the unknown boundary values c_j . The fourth line in (36) gives the system of N equations for N unknowns c_j . Since the boundary value problem for ζ is linear, we start with the estimates for the basis functions ζ_i that solve the problem

$$\begin{cases} -\Delta\zeta_i + \zeta_i = 0, & \text{in } \Omega_\delta, \\ \zeta_i = 0, & \text{on } \partial\Omega, \\ \zeta_i = \delta_{ij}, & \text{on } \partial\omega_\delta^j, \quad j = 1 \dots N, \end{cases} \quad (37)$$

for every $i = 1, \dots, N$. Then, using representation $\zeta = \sum_i c_i \zeta_i$, we will solve the linear system for c_i .

We use the method of sub- and supersolutions to get estimates for ζ_i . By the Maximum Principle, we have that $0 \leq \zeta_i \leq 1$ for every $i = 1, \dots, N$. In the case of a radially symmetric domain with one hole at the center, the solutions of (37) are the modified Bessel functions. We show that they provide a good approximation for ζ_i . First, fix $i \in 1 \dots N$ and construct a supersolution for ζ_i . Take $R_{\max} > 0$ such that $\Omega \in B(a^i, R_{\max})$ and set

$$\zeta_i^{\sup} = \frac{K_0\left(\frac{|x-a^i|}{R_{\max}}\right)}{K_0\left(\frac{\delta}{R_{\max}}\right)}. \quad (38)$$

The function ζ_i^{\sup} is strictly positive in Ω_δ , equals 1 on $\partial\omega_\delta^i$, and has $[-\Delta + I]\zeta_i^{\sup} = 0$. Therefore

it satisfies

$$\begin{cases} -\Delta \zeta_i^{\text{sup}} + \zeta_i^{\text{sup}} = 0 & \text{in } \Omega_\delta, \\ \zeta_i^{\text{sup}} > 0 & \text{on } \partial\Omega, \\ \zeta_i^{\text{sup}} = 1 & \text{in } \omega_\delta^i, \\ \zeta_i^{\text{sup}} > 0 & \text{in } \omega_\delta^j, j \neq i, \quad j = 1 \dots N, \end{cases} \quad (39)$$

and is thus a supersolution. This yields the bound

$$0 \leq \zeta_i \leq \zeta_i^{\text{sup}} \text{ in } \Omega, \quad i = 1 \dots N. \quad (40)$$

Next, we construct a subsolution. Take $R_{\min} > 0$ such that $B(a^i, 2R_{\min}) \in \Omega_\delta$ for every $i = 1 \dots N$ and set

$$\zeta_i^{\text{sub}} = \frac{K_0 \left(\frac{|x - a^i|}{R_{\min}} \right)}{K_0 \left(\frac{\delta}{R_{\min}} \right)} \quad (41)$$

The Bessel function is a fundamental solution of $[-\Delta + I]u = \delta(x)$ and it is decreasing, therefore ζ_i^{sub} is negative outside $B(a^i, R_{\min})$. Thus it satisfies

$$\begin{cases} -\Delta \zeta_i^{\text{sub}} + \zeta_i^{\text{sub}} = 0 & \text{in } \Omega_\delta, \\ \zeta_i^{\text{sub}} < 0 & \text{on } \partial\Omega, \\ \zeta_i^{\text{sub}} = 1 & \text{in } \omega_\delta^i, \\ \zeta_i^{\text{sub}} < 0 & \text{in } \omega_\delta^j, j \neq i, \quad j = 1 \dots N, \end{cases} \quad (42)$$

and is thus a subsolution. This, together with (40), implies that

$$\max(0, \zeta_i^{\text{sub}}) \leq \zeta_i \leq \zeta_i^{\text{sup}}, \quad (43)$$

for every $i = 1 \dots N$, giving a very sharp description of the behavior of ζ_i near i th hole. Note that, for $x \in \partial\omega_\delta^i$, we have

$$\frac{L_1}{\delta \log \delta} \leq \frac{\partial \zeta_i^{\text{sub}}}{\partial \nu}(x) \leq \frac{\partial \zeta_i^{\text{sup}}}{\partial \nu}(x) \leq \frac{L_2}{\delta \log \delta} \quad (44)$$

with $L_1, L_2 > 0$, therefore

$$\frac{\partial \zeta_i}{\partial \nu}(x) \sim \frac{1}{\delta \log \delta} \text{ on } \partial\omega_\delta^i. \quad (45)$$

To estimate the normal derivative of ζ_i on $\partial\omega_\delta^j$ for $j \neq i$ we need a better supersolution that captures the appropriate Dirichlet boundary conditions. Outside of $B(a^i, R_{\min})$, we have

$$|\zeta_i(x)| \leq \frac{K_0 \left(\frac{R_{\min}}{R_{\max}} \right)}{K_0 \left(\frac{\delta}{R_{\max}} \right)} \leq C_R |\log \delta|^{-1}. \quad (46)$$

Construct ζ_{ij}^{sup} that solves the following conditions:

$$\begin{cases} -\Delta \zeta_{ij}^{\text{sup}} + \zeta_{ij}^{\text{sup}} = 0 & \text{in } B(a^j, R_{\min}) \setminus \overline{B(a^j, \delta)}, \\ \zeta_{ij}^{\text{sup}} = C_R |\log \delta|^{-1} & \text{on } \partial B(a^j, R_{\min}), \\ \zeta_{ij}^{\text{sup}} = 0 & \text{on } \partial\omega_\delta^j. \end{cases} \quad (47)$$

This problem is radially symmetric in $B(a^j, R_{\min}) \setminus \overline{B(a^j, \delta)}$. The function

$$\zeta_{ij}^{\sup} = C_1 I_0(r) + C_2 K_0(r), \quad r = |x - a^j| \quad (48)$$

with

$$C_1 \sim -|\log \delta|^{-1} \quad \text{and} \quad C_2 \sim |\log \delta|^{-2}. \quad (49)$$

satisfies (47) because the modified Bessel functions I_0 and K_0 behave as 1 and $-\log r$, respectively, near the origin. Therefore

$$0 \leq \frac{\partial \zeta_i}{\partial \nu} \leq \frac{\partial \zeta_{ij}^{\sup}}{\partial \nu} = \frac{C_{ij}}{\delta |\log \delta|^2} \text{ on } \partial \omega_\delta^j. \quad (50)$$

As a result

$$\int_{\partial \omega_\delta^j} \left| \frac{\partial \zeta_i}{\partial \nu} \right| ds \leq \frac{C}{|\log \delta|^2}. \quad (51)$$

for all $i \neq j$. Combining the estimates on the behavior of ζ_i on $\partial \omega_\delta^i$ in (45) with (51) and estimating the constants c_i using the fourth equation in (36) we find:

$$\begin{aligned} \pi \delta^2 |c_i| + |A_i^\delta| &\geq \left| \int_{\partial \omega_\delta^i} \frac{\partial \zeta_i}{\partial \nu} ds \right| \geq \left| c_i \int_{\partial \omega_\delta^i} \frac{\partial \zeta_i}{\partial \nu} ds \right| - \sum_{j \neq i} \left| c_j \int_{\partial \omega_\delta^i} \frac{\partial \zeta_j}{\partial \nu} ds \right| \\ &\geq |c_i| \frac{C_1}{|\log \delta|} - \sum_{j \neq i}^N |c_j| \frac{C_2}{|\log \delta|^2} \end{aligned} \quad (52)$$

or

$$|c_i| \left(\frac{C_1}{|\log \delta|} - \pi \delta^2 \right) - \sum_{j \neq i}^N |c_j| \frac{C_2}{|\log \delta|^2} \leq |A_i^\delta|, \quad (53)$$

with some positive $C_1, C_2 > 0$ for all $i = 1 \dots N$. The coefficient matrix is a small perturbation of the identity matrix, up to the factor $C_1 |\log \delta|^{-1}$. This allows us to conclude that

$$|c_i| \leq |D| O(\delta \log \delta) + O(\delta \log^2 \delta) \quad (54)$$

for all $i = 1 \dots N$. Let

$$c_i = \max_j |c_j|. \quad (55)$$

Then

$$|c_i| \leq |A_i^\delta| \left(\frac{C_1}{|\log \delta|} - \pi \delta^2 - (N-1) \frac{C_2}{|\log \delta|^2} \right)^{-1} \leq |D| O(\delta \log \delta) + O(\delta \log^2 \delta), \quad (56)$$

hence

$$\|\zeta\|_{L^\infty(\Omega_\delta)} \leq \sum_j |c_j| \leq C_1 |D| \delta |\log \delta| + C_2 \delta |\log \delta|^2. \quad (57)$$

The statement of the lemma for

$$h_3 = \eta_0 + \sum_{j=1}^N D^j \eta_j + \sum_{j=1}^N c_j \zeta_j \quad (58)$$

then follows once we combine the estimates above. \square

Proof of Theorem 2, continued. We are now able to find the asymptotics for the energy $l_\delta(D) = GL_\delta[h_{\delta D}]$:

$$\begin{aligned}
l_\delta(D) &= GL_\delta[h_1 + h_2 + h_3] \\
&= \frac{1}{2} \int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2} \int_{\Omega_\delta} |\nabla h_2|^2 dx + \frac{1}{2} \int_{\Omega_\delta} |\nabla h_3|^2 dx \\
&\quad + \frac{1}{2} \int_{\Omega} (h_1 - h_{ext})^2 dx + \frac{1}{2} \int_{\Omega} h_2^2 dx + \frac{1}{2} \int_{\Omega_\delta} h_3^2 dx \\
&\quad + \int_{\Omega_\delta} [\nabla(h_1 - h_{ext}) \cdot \nabla \hat{h} + (h_1 - h_{ext}) \hat{h}] dx + \int_{\Omega_\delta} [\nabla h_2 \cdot \nabla h_3 + h_2 h_3] dx \\
&\quad + |D|^2 O(\delta^2 |\log \delta|^3) + O(\delta^2 |\log \delta|^3),
\end{aligned} \tag{59}$$

where $\hat{h} = h_2 + h_3$ and the integrals over holes ω_δ^j are the source of the error. Next, we estimate each term in (59). The terms that involve h_1 only do not depend on the degrees of the hole vortices and thus they do not play a role in the minimization of $l_\delta(D)$:

$$\begin{aligned}
\frac{1}{2} \int_{\Omega_\delta} |\nabla h_1|^2 dx + \frac{1}{2} \int_{\Omega} (h_1 - h_{ext})^2 dx &= h_{ext}^2 \frac{1}{2} \int_{\Omega_\delta} |\nabla \xi_0|^2 dx + h_{ext}^2 \frac{1}{2} \int_{\Omega} (1 - \xi_0)^2 dx \\
&= O(|\log \delta|^2).
\end{aligned} \tag{60}$$

The gradient of h_2 gives the main quadratic term:

$$\begin{aligned}
\frac{1}{2} \int_{\Omega_\delta} |\nabla h_2|^2 dx &= \frac{1}{2} \sum_{j=1}^N (D^j)^2 \int_{T_j} |\nabla(\theta_j(x) \phi_j(x))|^2 dx \\
&= \pi \sum_{j=1}^N (D^j)^2 \left[\int_{\delta}^{R/4} |K_0(r)'|^2 r dr + \int_{R/4}^R \left| \frac{d}{dr} (K_0(r) \phi(r)) \right|^2 r dr \right] \\
&= \pi \sum_{j=1}^N (D^j)^2 \left[\int_{\delta}^{R/4} \left| -\frac{1}{r} + O(r \log r) \right|^2 r dr + O(1) \right]
\end{aligned} \tag{61}$$

$$= \pi \sum_{j=1}^N (D^j)^2 |\log \delta| + |D|^2 O(1). \tag{62}$$

The L^2 -norm of h_2 is much smaller, indeed:

$$\frac{1}{2} \int_{\Omega} h_2^2 dx = \pi \sum_{j=1}^N (D^j)^2 \int_0^{R/2} |\theta_j \phi|^2 r dr = |D|^2 O(1). \tag{63}$$

We now estimate the integral involving \hat{h} that gives the linear terms in terms of the degrees. Note that, since $h_{\delta D}$ and h_1 solve the homogeneous equation $[-\Delta + I] h = 0$, then so does their difference

$$\widehat{h} = h_{\delta D} - h_1:$$

$$\begin{aligned}
\langle h_1 - h_{ext}, \widehat{h} \rangle_{H^1(\Omega_\delta)} &= \int_{\Omega_\delta} (h_1 - h_{ext}) \left(-\Delta \widehat{h} + \widehat{h} \right) dx - \int_{\partial\Omega_\delta} (h_1 - h_{ext}) \frac{\partial \widehat{h}}{\partial \nu} ds \\
&= \sum_{j=1}^N \int_{\partial\omega_\delta^j} (h_1 - h_{ext}) \frac{\partial(h_2 + h_3)}{\partial \nu} ds \\
&= \sum_{j=1}^N \int_{\partial\omega_\delta^j} (h_1 - h_{ext}) \left[D^j \left(\frac{1}{\delta} + O(\delta \log \delta) \right) + O(\log \delta) + |D|O(1) \right] ds \\
&= \sum_{j=1}^N D^j (h_1(a^j) - h_{ext}) 2\pi\delta \cdot \frac{1}{\delta} + O(\delta |\log \delta|^2) + |D|O(\delta \log \delta) \\
&= -2\pi\sigma |\log \delta| \sum_{j=1}^N D^j (1 - \xi_0(a^j)) + O(\delta |\log \delta|^2) + |D|O(\delta \log \delta), \tag{64}
\end{aligned}$$

where use the notation $\langle u, v \rangle_{H^1} = \int [\nabla u \cdot \nabla v + uv] dx$. The other terms in (59) are small and are estimated using integration by parts:

$$\begin{aligned}
\|h_3\|_{H^1(\Omega_\delta)}^2 &= \int_{\Omega_\delta} h_3 (-\Delta h_3 + h_3) dx - \int_{\partial\Omega_\delta} h_3 \frac{\partial h_3}{\partial \nu} ds = \sum_{j=1}^N \int_{\partial\omega_\delta^j} h_3 \frac{\partial h_3}{\partial \nu} ds \\
&= C\delta (C_1\delta |\log \delta|^2 + C_2|D|) (C_1|\log \delta| + C_2|D|) \\
&= O(\delta^2 |\log \delta|^3) + |D|^2 O(\delta |\log \delta|) \tag{65}
\end{aligned}$$

$$\begin{aligned}
\langle h_2, h_3 \rangle_{H^1(\Omega_\delta)} &= \int_{\Omega_\delta} h_2 (-\Delta h_3 + h_3) dx - \int_{\partial\Omega_\delta} h_2 \frac{\partial h_3}{\partial \nu} ds = \sum_{j=1}^N \int_{\partial\omega_\delta^j} h_2 \frac{\partial h_3}{\partial \nu} ds \\
&= \sum_{j=1}^N 2\pi\delta D^j K_0(\delta) (C_1|\log \delta| + C_2|D|) \\
&= |D|^2 O(\delta |\log \delta|^2) \tag{66}
\end{aligned}$$

Combining all of the above estimates, we obtain the asymptotic expansion (17). \square

Corollary 1. The leading part of the energy $l_\delta(Z)$ is a sum of N one-dimensional parabolas with the vertices at

$$Z_j = \sigma(1 - \xi_0(a^j)) \in \mathbb{R}. \tag{67}$$

Since the degrees are integer-valued, the minimizing degrees D^j are the integers, closest to Z_j :

$$D^j = \llbracket \sigma(1 - \xi_0(a^j)) \rrbracket, \tag{68}$$

where $\llbracket x \rrbracket$ denotes the integer nearest to x .

4 Energy Decomposition

Since $(u_{\delta D}, A_{\delta D})$ is an admissible pair for the problem (5), we can use the representation of S^1 -valued energy (17) with $D = 0$ to obtain an upper bound

$$GL_{\delta}^{\varepsilon}[u_{\delta}^{\varepsilon}, A_{\delta}^{\varepsilon}] \leq GL_{\delta}^{\varepsilon}[u_{\delta 0}, A_{\delta 0}] \leq C|\log \delta|^2 \quad (69)$$

on the energy of the minimizer of (5). In order to obtain a matching lower energy bound, we need to localize the regions of the domain where the magnitude of the order parameter is small. To this end, we use the following theorem.

Theorem 3 (Ball Construction Method [19]). *For any $\alpha \in (0, 1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if (u, A) is a configuration such that $GL_{\delta}^{\varepsilon}[u, A] < \varepsilon^{\alpha-1}$, where ε is an inverse of the Ginzburg-Landau parameter, the following holds.*

For any $1 > \rho > C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite collection of disjoint closed balls $\mathfrak{B} = \{B_i = B(b^i, r_i)\}_{i \in \mathfrak{J}}$ such that

1. $r(\mathfrak{B}) = \rho$ where $r(\mathfrak{B}) = \sum_{i \in \mathfrak{J}} r(B_i)$.

2. Letting $V = \Omega_{\delta} \cap \cup_{i \in \mathfrak{J}} B_i$,

$$\left\{x \in \Omega_{\delta} \mid ||u(x)| - 1| \geq \varepsilon^{\alpha/4}\right\} \subset V. \quad (70)$$

3. Writing $d_i = \deg(u, \partial B_i)$, if $B_i \subset \Omega_{\delta}$ and $d_i = 0$ otherwise,

$$\frac{1}{2} \int_V \left[|\nabla_A u|^2 + \rho^2 |\operatorname{curl} A|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right] dx \geq \pi d \left(\log \frac{\rho}{d\varepsilon} - C \right), \quad (71)$$

where $d = \sum_{i \in \mathfrak{J}} |d_i|$ is assumed to be nonzero and C is a universal constant.

4. There exists a universal constant C such that

$$d \leq C \frac{GL_{\delta}^{\varepsilon}[u, A]}{\alpha |\log \varepsilon|}. \quad (72)$$

We consider now a domain with N holes $\omega_{\delta}^j = B(a^j, \delta)$ so that $\Omega_{\delta} = \Omega \setminus \cup_{j=1}^N \overline{\omega_{\delta}^j}$. Set $\alpha = 1/2$ and $\rho = \delta^2/2$ in the ball construction method. Assume that ε is small enough so that $|u(x)| > 1 - \theta$ on $\Omega_{\delta} \cap (\cup_{i \in \mathfrak{J}} B_i)$. The parameter θ will be chosen later, in Section 6.

Lemma 2. *Let $(u_{\delta}^{\varepsilon}, A_{\delta}^{\varepsilon})$ be a minimizer of the problem (5). Then the following energy decomposition holds:*

$$GL_{\delta}^{\varepsilon}[u_{\delta}^{\varepsilon}, A_{\delta}^{\varepsilon}] = GL_{\delta}[u_{\delta D}, A_{\delta D}] + F_{\delta}[v, B] - \int_{\Omega_{\delta}} \nabla^{\perp} h_{\delta D} \cdot \operatorname{Im} \overline{v} \nabla v \, dx + o(1) \quad (73)$$

where $u_{\delta}^{\varepsilon} = v u_{\delta D}$, $A_{\delta}^{\varepsilon} = A_{\delta D} + B$, $h_{\delta D} = \operatorname{curl} A_{\delta D}$ and

$$F_{\delta}[v, B] = \frac{1}{2} \int_{\Omega_{\delta}} \left(|(\nabla - iB)v|^2 + \frac{1}{2\varepsilon^2} (1 - |v|^2)^2 \right) dx + \frac{1}{2} \int_{\Omega} (\operatorname{curl} B)^2 \, dx. \quad (74)$$

Here $(u_{\delta D}, A_{\delta D})$ is the minimizer of the S^1 -valued problem (15) with the prescribed degrees D .

Proof. Using the representation (20) of Ginzburg-Landau functional in terms of $h_{\delta D}$, note that the pair $(u_{\delta D}, A_{\delta D})$ satisfies the following equation

$$\nabla^\perp h_{\delta D} = -\text{Im} (\bar{u}_{\delta D} \nabla u_{\delta D} - i A_{\delta D}) \quad (75)$$

outside of the holes. We start the proof with representing $GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon]$ as a sum of three terms:

$$GL_\delta[u_\delta^\varepsilon, A_\delta^\varepsilon] = I_1 + I_2 + I_3, \quad (76)$$

where

$$I_1 = \frac{1}{2} \int_\Omega |\nabla u_\delta^\varepsilon - i A_\delta^\varepsilon u_\delta^\varepsilon|^2 dx, \quad I_2 = \frac{1}{4\varepsilon^2} \int_{\Omega_\delta} (1 - |u_\delta^\varepsilon|^2)^2 dx, \quad I_3 = \frac{1}{2} \int_\Omega (\text{curl } A_\delta^\varepsilon - h_{ext})^2 dx. \quad (77)$$

Observe that $|u_\delta^\varepsilon| = |v|$ as $u_\delta^\varepsilon = v u_{\delta D}$ and $|u_{\delta D}| = 1$. Hence we can rewrite I_2 as

$$I_2 = \frac{1}{4\varepsilon^2} \int_{\Omega_\delta} (1 - |u_{\delta D}|^2)^2 dx = \frac{1}{4\varepsilon^2} \int_{\Omega_\delta} (1 - |v|^2)^2 dx, \quad (78)$$

giving us the second term in the definition of $F_\delta[v, B]$. Now rewrite I_3 :

$$\begin{aligned} I_3 &= \frac{1}{2} \int_\Omega (\text{curl } A_\delta^\varepsilon - h_{ext})^2 dx \\ &= \frac{1}{2} \int_\Omega (h_{\delta D} - h_{ext})^2 dx + \frac{1}{2} \int_\Omega (\text{curl } B)^2 dx + \int_\Omega \text{curl } B \cdot (h_{\delta D} - h_{ext}) dx \end{aligned} \quad (79)$$

Here, the first term is a part of $GL_\delta[u_{\delta D}, A_{\delta D}]$ and the second term is a part of $F_\delta[v, B]$. The last term will eventually cancel with a component of I_1 . To this end,

$$\begin{aligned} |\nabla u_\delta^\varepsilon - i A_\delta^\varepsilon u_\delta^\varepsilon|^2 &= |v (\nabla u_{\delta D} - i A_{\delta D} u_{\delta D}) + u_{\delta D} (\nabla v - i B v)|^2 \\ &= |v|^2 |\nabla u_{\delta D} - i A_{\delta D} u_{\delta D}|^2 + |u_{\delta D}|^2 |\nabla v - i B v|^2 \\ &\quad + 2 \text{Re} (\bar{u}_{\delta D} (\nabla u_{\delta D} - i A_{\delta D} u_{\delta D}) \cdot v (\nabla \bar{v} + i B \bar{v})) \\ &= |\nabla v - i B v|^2 + |v|^2 |\nabla h_{\delta D}|^2 \\ &\quad + 2|v|^2 \nabla^\perp h_{\delta D} \cdot B - 2 \nabla^\perp h_{\delta D} \cdot \text{Im} (\bar{v} \nabla v) \end{aligned} \quad (80)$$

The first term in (80) contributes to $F_\delta[v, B]$. The last term is included in the right hand side of the decomposition. The sum of two other terms has the form $|v|^2 \cdot R(x)$, where

$$R(x) = |\nabla h_{\delta D}|^2 + 2 \nabla^\perp h_{\delta D} \cdot B$$

Now add and subtract $\frac{1}{2} \int_{\Omega_\delta} R(x) dx$ to the energy $GL_\delta[u_\delta^\varepsilon, A_\delta^\varepsilon]$. The first term $\frac{1}{2} \int_{\Omega_\delta} |\nabla h_{\delta D}|^2 dx$ is a part of $GL_\delta[u_{\delta D}, A_{\delta D}]$. Using integration by parts we prove that the second term $\int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot B dx$

indeed cancels with the last term in the representation (79) of I_3 as alluded to above:

$$\begin{aligned}
\int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot B \, dx &= \int_{\Omega_\delta} \nabla^\perp (h_{\delta D} - h_{ext}) \cdot B \, dx \\
&= \int_{\partial\Omega_\delta} (h_\delta - h_{ext}) B \cdot \tau \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{ext}) \nabla^\perp \cdot B \, dx \\
&= - \sum_{j=1}^N (h_{\delta D} - h_{ext})|_{\partial B(a^j, \delta)} \int_{\partial B(a^j, \delta)} B \cdot \tau \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{ext}) \operatorname{curl} B \, dx \\
&= - \sum_{j=1}^N (h_{\delta D} - h_{ext})|_{\partial B(a^j, \delta)} \int_{B(a^j, \delta)} \operatorname{curl} B \, dS - \int_{\Omega_\delta} (h_{\delta D} - h_{ext}) \operatorname{curl} B \, dx \\
&= - \int_{\Omega} (h_{\delta D} - h_{ext}) \operatorname{curl} B \, dx.
\end{aligned} \tag{81}$$

Here we used the facts that $h_{\delta D} = h_{ext}$ on the boundary $\partial\Omega$ and $h_{\delta D} = \text{const}$ in $B(a^j, \delta)$ that follow from the equation for $h_{\delta D}$.

Adding up the results above gives:

$$\begin{aligned}
GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon] &= GL_\delta[u_{\delta D}, A_{\delta D}] + F_\delta[v, B] \\
&\quad - \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{v} \nabla v \, dx + \int_{\Omega_\delta} (1 - |v|^2) R(x) \, dx + o(1)
\end{aligned} \tag{82}$$

The remaining task is to show that

$$I = \int_{\Omega_\delta} (1 - |v|^2) R(x) \, dx$$

goes to zero as $\delta \rightarrow 0$. Hölder's inequality implies that

$$|I| \leq \|1 - |v|^2\|_{L^2(\Omega_\delta)} \cdot \left(2\|\nabla h_{\delta D}\|_{L^4(\Omega_\delta)}^2 + \|B\|_{L^4(\Omega_\delta)}^2 \right). \tag{83}$$

The first multiplier in this expression is less than $M\varepsilon |\log \delta|$ when $\delta \rightarrow 0$ because of the a priori estimate on the energy. Using the relation between ε and δ

$$|\log \varepsilon| \gg |\log \delta|, \tag{84}$$

we show that ε is sufficiently small to compensate for the growth of the other terms.

The function $h_{\delta D}$ is described in Theorem 2 and because of Lemma 6 it satisfies the estimate

$$\|\nabla h_{\delta D}\|_{L^4(\Omega_\delta)}^2 \leq \frac{C|\log \delta|^2}{\delta^2}. \tag{85}$$

In order to estimate $\|B\|_{L^4(\Omega_\delta)}$, recall that $\operatorname{div} A_\delta^\varepsilon = 0$ due to the gauge invariance. Then by the Poincaré's lemma A_δ^ε has a potential, i.e. there exists Π_δ^ε such that $\nabla^\perp \Pi_\delta^\varepsilon = A_\delta^\varepsilon$. Substituting this into $h_\delta^\varepsilon = \operatorname{curl} A_\delta^\varepsilon$, we obtain the equality $\Delta \Pi_\delta^\varepsilon = h_\delta^\varepsilon$. The function Π_δ^ε is a potential so we are able to make it zero on the boundary $\partial\Omega$. From the theory of elliptic operators and the a priori energy estimate, we obtain

$$\|\Pi_\delta^\varepsilon\|_{H^2(\Omega)}^2 \leq \|h_\delta^\varepsilon\|_{L^2(\Omega)}^2 \leq C|\log \delta|^2. \tag{86}$$

Since the embedding $H^1(\Omega) \subset L^4(\Omega)$ is continuous we have

$$\|A_\delta^\varepsilon\|_{L^4(\Omega_\delta)} \leq C\|\Pi_\delta^\varepsilon\|_{H^2(\Omega)} \leq C|\log \delta|.$$

The same estimate holds for $A_{\delta D}$. Using the decomposition $A_\delta^\varepsilon = B + A_{\delta D}$ we obtain this estimate for B :

$$\|B\|_{L^4(\Omega_\delta)} \leq C|\log \delta|$$

Combining all estimates obtained in this section, we conclude that

$$|I| \leq C\varepsilon|\log \delta| \left(\frac{|\log \delta|^2}{\delta^2} + |\log \delta|^2 \right).$$

The condition $|\log \varepsilon| \gg |\log \delta|$ implies that ε is much smaller than any power of δ , therefore I goes to zero as $\delta \rightarrow 0$ that completes the proof. \square

5 Absence of Bulk Vortices

In this section we further analyze the energy decomposition (73). The energy of the unconstrained solution is minimal, hence

$$GL_\delta^\varepsilon[u_\delta^\varepsilon, A_\delta^\varepsilon] \leq GL_\delta[u_{\delta D}, A_{\delta D}], \quad (87)$$

and using (73) we have

$$F_\delta[v, B] \leq \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{v} \nabla v \, dx + o(1). \quad (88)$$

First, we derive an upper bound for the integral term in (88) and thus for the energy F_δ . We start with a simple fact that will also be used later on.

Proposition 1. *Given a sufficiently smooth domain $S \subset \mathbb{R}^2$ and any $R \in L^2(S, \mathbb{R})$, $P \in H^1(S, \mathbb{R}^2)$, and $v \in H^1(S, \mathbb{C})$ such that $|v| \leq 1$ a.e. $x \in S$, we have that*

$$\begin{aligned} \left| \int_S R(x) \cdot \operatorname{Im} \bar{v} \nabla v \, dx \right| &\leq \left| \int_S R(x) \cdot (\operatorname{Im} \bar{v} (\nabla - iP)v + P|v|^2) \, dx \right| \\ &\leq \|R\|_{L^2(S)} \cdot (\|(\nabla - iP)v\|_{L^2(S)} + \|P\|_{L^2(S)}) \end{aligned} \quad (89)$$

We are now in the position to state and prove

Lemma 3. *The following estimates hold:*

$$F_\delta[v, B] \leq |\log \delta|^2, \quad (90)$$

$$\left| \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{v} \nabla v \, dx \right| \leq |\log \delta|^2. \quad (91)$$

Proof. Use (89) and Poincaré inequality to estimate the integral term in (88):

$$\begin{aligned} \left| \int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{v} \nabla v \, dx \right| &\leq \|\nabla h_{\delta D}\|_{L^2(\Omega_\delta)} \cdot (\|(\nabla - iB)v\|_{L^2(\Omega_\delta)} + C_\Omega \|\operatorname{curl} B\|_{L^2(\Omega)}) \\ &\leq \frac{1}{2\alpha} \|\nabla h_{\delta D}\|_{L^2(\Omega_\delta)}^2 + \frac{\alpha}{2} (\|(\nabla - iB)v\|_{L^2(\Omega_\delta)}^2 + C_\Omega^2 \|\operatorname{curl} B\|_{L^2(\Omega)}^2) \\ &\leq O(|\log \delta|^2) + \frac{1}{2} F_\delta[v, B] \end{aligned} \quad (92)$$

where $\alpha = \min(1, C_\Omega^{-2})$. Here we have used the standard fact that $|u_\delta^\varepsilon| \leq 1$ and, therefore, $|v| \leq 1$ a.e. $x \in \Omega_\delta$.

Combining the inequality (92) with (88) gives

$$F_\delta[v, B] \leq O(|\log \delta|^2). \quad (93)$$

The estimates (92) and (93) imply (91). \square

The bound (93) allows us to apply the ball construction method to F_δ . Theorem 3 gives the following lower bound on the energy inside “bad” disks:

$$F_\delta[v, B; B_i] \geq \pi |d_i| \left(\log \frac{\delta^2}{|d_i| \varepsilon} - C \right) \text{ for every } i \in \mathfrak{I}. \quad (94)$$

Here $F_\delta[v, B; B_i]$ is the energy $F_\delta[v, B]$ where first two integrals are taken over the domain $B_i = B(b^i, r_i)$. To continue working with (88) we prove the following lemma.

Lemma 4. *The following representation holds:*

$$\int_{\Omega_\delta} \nabla^\perp h_{\delta D} \cdot \text{Im } \bar{v} \nabla v \, dx = 2\pi \sum_{i \in \mathfrak{I}_1} (h_{ext} - h_{\delta D}(b^i)) d_i + 2\pi \sum_{j=1}^N D_v^j (h_{ext} - H_R^j) + O(1) \quad (95)$$

where $D_v^j = \deg(v, \gamma_r^j) = D_{\delta, \varepsilon}^j - D^j$, the circular curves $\gamma_r^j = \partial B(a^j, R)$ enclose ω_δ^j with $R = \delta + O(\delta^2)$, the quantities $H_R^j = D^j K_0(R) + h_{ext} \xi_0(a^j)$, and \mathfrak{I}_1 includes only the balls that are proper subsets of $\Omega_\delta \setminus \cup_{j=1}^N \bar{\omega}_\delta^j$ and do not intersect the boundary $\partial \Omega_\delta$.

Proof. We divide the domain Ω_δ into three disjoint parts:

$$\Omega_\delta = S \cup V \cup G, \quad (96)$$

where $S = \cup_{j=1}^N S_j$ consists of the annuli between $\partial \omega_\delta^j$ and γ_r^j , the set $V = [(\cup_{i \in \mathfrak{I}} B_i) \setminus S] \cap \Omega_\delta$ consists of the “bad” disks, and G corresponds to the remainder of the set Ω_δ .

Consider the subdomains S , V , and G separately. The balls B_i —as well as stripes S_j —are very small so that

$$\begin{aligned} \int_{V \cup S} \nabla^\perp h_{\delta D} \cdot \text{Im } \bar{v} \nabla v \, dx &\leq \text{meas}(V \cup S)^{1/4} \cdot \|\nabla^\perp h_{\delta D}\|_{L^4(V \cup S)} \\ &\quad \cdot (\|(\nabla - iB)v\|_{L^2(V \cup S)} + \|B\|_{L^2(V \cup S)}) \\ &\leq C\delta^{3/4} \cdot |\log \delta| \cdot |\log \delta| = o(1). \end{aligned} \quad (97)$$

Introduce the function $w = v/|v|$. Then

$$\begin{aligned} \int_G \nabla^\perp h_{\delta D} \cdot \text{Im } \bar{v} \nabla v \, dx &= \int_G \nabla^\perp h_{\delta D} \cdot \text{Im } \bar{w} \nabla w \, dx + \int_G \nabla^\perp h_{\delta D} \cdot (\text{Im } \bar{v} \nabla v - \text{Im } \bar{w} \nabla w) \, dx \\ &= I_1 + I_2. \end{aligned} \quad (98)$$

To estimate the second integral, use the following:

$$\begin{aligned} \text{Im } \bar{v} \nabla v - \text{Im } \bar{w} \nabla w &= \text{Im } (\bar{w}|v|(w\nabla|v| + |v|\nabla w) - \bar{w}\nabla w) \\ &= \text{Im } (|v|\nabla|v| + (|v|^2 - 1)\bar{w}\nabla w) = (|v|^2 - 1)\text{Im } \bar{w}\nabla w \end{aligned} \quad (99)$$

and

$$|\nabla v|^2 = |v|^2 |\nabla w|^2 + |\nabla |v||^2 \geq (1 - \theta)^2 |\nabla w|^2 \geq \frac{1}{4} |\nabla w|^2. \quad (100)$$

since by Theorem 3 we have $|v| \geq 1 - \theta$ outside B_i . The function v admits the same estimate as u_δ^ε . Add and subtract iBv to get

$$\begin{aligned} \frac{1}{2} \|\nabla v\|_{L^2(G)}^2 &= \frac{1}{2} \int_G |\nabla v|^2 dx \leq \int_G (|(\nabla - iB)v|^2 + |v|^2 |B|^2) dx \\ &\leq \int_{\Omega_\delta} |(\nabla - iB)v|^2 dx + C_\Omega \int_\Omega |\operatorname{curl} B|^2 dx \leq C |\log \delta|^2. \end{aligned} \quad (101)$$

This leads to the following estimate:

$$\begin{aligned} |I_2| &\leq \int_G \nabla^\perp h_{\delta D} \cdot (|v|^2 - 1) \operatorname{Im} \bar{w} \nabla w dx \\ &\leq \|\nabla^\perp h_{\delta D}\|_{L^\infty(G)} \cdot \int_G (|v|^2 - 1) \cdot |\nabla w| dx \\ &\leq C \delta^{-1} \cdot \int_G (|v|^2 - 1) \cdot 2 |\nabla v| dx \\ &\leq C \delta^{-1} \cdot \| |v|^2 - 1 \|_{L^2(G)} \cdot \|\nabla v\|_{L^2(G)} \\ &\leq C \delta^{-1} \cdot \varepsilon |\log \delta| \cdot |\log \delta| = o(1) \end{aligned} \quad (102)$$

due to (9).

Now rewrite the integral I_1 . Integrating by parts, we obtain:

$$\begin{aligned} I_1 &= \int_G \nabla^\perp (h_{\delta D} - h_{ext}) \cdot \operatorname{Im} \bar{w} \nabla w dx = - \int_G (h_{\delta D} - h_{ext}) \nabla^\perp \cdot \operatorname{Im} \bar{w} \nabla w dx \\ &\quad + \int_{\partial \Omega} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds - \int_{\partial V} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds \\ &\quad - \int_{\cup_j \gamma_j} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds \\ &= - \sum_{i \in \mathfrak{J}} I_{1i} - \sum_{j=1}^N \int_{\gamma_r^j} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds \end{aligned} \quad (103)$$

where $I_{1i} = \int_{\partial V_i} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds$ and $V_i = B_i \cap \Omega_\delta$. The term $\nabla^\perp \cdot \operatorname{Im} \bar{w} \nabla w = \operatorname{curl} \nabla \Phi = 0$, where Φ is a phase of w , disappears.

Since the curves γ_r^j are small, we can approximate $h_{\delta D}$ by a constant H_R^j to conclude that

$$\int_{\gamma_r^j} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds = 2\pi D_v^j (H_R^j - h_{ext}) + \int_{\gamma_r^j} (h_{\delta D} - H_R^j) \operatorname{Im} \bar{w} \nabla w \cdot \tau ds.$$

Set $H_R^j = h_{ext} \xi_0(a^j) + D^j K_0(R)$. Using the decomposition (21) of $h_{\delta D}$, we get

$$|h_{\delta D}(x) - H_R^j| \leq h_{ext} |\xi_0(x) - \xi_0(a^j)| + |h_3(x)| \leq C_1 \delta |\log \delta|^2 + C_2 |D| \quad (104)$$

for $x \in \gamma_r^j$. This yields

$$\left| \int_{\gamma_r^j} (h_{\delta D} - H_R^j) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds \right| \leq (C_1 \delta |\log \delta|^2 + C_2 |D|) \cdot D_v^j = O(1). \quad (105)$$

As a result we estimate that

$$I_1 = - \sum_{i \in \mathfrak{J}} I_{1i} - \sum_{j=1}^N 2\pi D_v^j (H_R^j - h_{ext}) + O(1). \quad (106)$$

We now consider two cases. First, suppose that the set $\mathfrak{J}_1 \subset \mathfrak{J}$ is such that $B_i \subset \Omega_\delta \setminus S$ for $i \in \mathfrak{J}_1$. We estimate the integrals I_{1i} in a similar way as we did for the hole vortices. Approximate $h_{\delta D}(x)$ by a constant value in the center of B_i :

$$I_{1i} = \int_{\partial V_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds + \int_{\partial V_i} (h_{\delta D}(b^i) - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds = J_{1i} + J_{2i} \quad (107)$$

Second integral directly gives the degree d_i of the possible bulk vortex:

$$J_{2i} = 2\pi d_i (h_{\delta D}(b^i) - h_{ext}). \quad (108)$$

To estimate J_{1i} we introduce the subdomains $U_i = V_i \cap \{x \mid |v(x)| \leq 1/2\}$ so that their boundaries are the level sets of v . We add and subtract the integral over ∂U_i :

$$\sum_{i \in \mathfrak{J}} J_{1i} = J_1 + J_2, \quad (109)$$

where

$$J_1 = \int_{\cup_{i \in \mathfrak{J}_1} \partial U_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds, \quad (110)$$

$$\begin{aligned} J_2 &= \int_{\cup_{i \in \mathfrak{J}_1} \partial V_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds - \int_{\cup_{i \in \mathfrak{J}_1} \partial U_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds \\ &= \int_{\cup_{i \in \mathfrak{J}_1} (V_i \setminus U_i)} \nabla^\perp \cdot [(h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w] \, dx = \int_{\cup_{i \in \mathfrak{J}_1} (V_i \setminus U_i)} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{w} \nabla w \, dx, \end{aligned} \quad (111)$$

since $\nabla^\perp \cdot \operatorname{Im} \bar{w} \nabla w = 0$. The term J_2 is small:

$$|J_2| \leq \operatorname{meas}(\mathfrak{B})^{1/2} \cdot \|\nabla^\perp h_{\delta D}\|_{L^\infty(\mathfrak{B})} \cdot 2\|\nabla v\|_{L^2(\mathfrak{B})} \leq O(\delta^2) \cdot O\left(\frac{1}{\delta}\right) \cdot O(|\log \delta|) = o(1). \quad (112)$$

To estimate J_1 , note, that $|v| = 1/2$ on ∂U_i so that $\nabla w \cdot \tau = 2\nabla v \cdot \tau$ on ∂U_i and:

$$\begin{aligned} J_1 &= \int_{\cup_{i \in \mathfrak{J}_1} \partial U_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds = 4 \int_{\cup_{i \in \mathfrak{J}_1} \partial U_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} \bar{v} \nabla v \cdot \tau \, ds \\ &= 4 \int_{\cup_{i \in \mathfrak{J}_1} U_i} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{v} \nabla v \, dx + 4 \int_{\cup_{i \in \mathfrak{J}_1} U_i} (h_{\delta D} - h_{\delta D}(b^i)) \operatorname{Im} (\nabla^\perp \bar{v} \cdot \nabla v) \, dx = L_1 + L_2. \end{aligned}$$

The first integral L_1 admits the same estimate as in (112). To estimate L_2 note that

$$|\operatorname{Im}(\nabla^\perp \bar{v} \cdot \nabla v)| \leq |\nabla^\perp \bar{v}| \cdot |\nabla v| = |\nabla v|^2. \quad (113)$$

Then

$$\begin{aligned} |L_2| &\leq 4 \sum_{i \in \mathcal{J}_1} \|h_{\delta D} - h_{\delta D}(b^i)\|_{L^\infty(U_i)} \cdot \|\nabla v\|_{L^2(\Omega)}^2 \\ &\leq 4 \sum_{i \in \mathcal{J}_1} \|\nabla h_{\delta D}\|_{L^\infty(U_i)} \cdot r_i \cdot |\log \delta|^2 \leq O\left(\frac{1}{\delta}\right) \cdot \delta^2 \cdot |\log \delta|^2 = o(1). \end{aligned} \quad (114)$$

Thus all integrals L_1 , L_2 , and therefore J_1 , J_2 , and J_{1i} are small. The only ingredient left to consider is the set \mathcal{J}_2 consisting of the balls that intersect the boundary $\partial\Omega$. Here the estimates are very similar to those on the balls from \mathcal{J}_1 if we recall the boundary condition $h_{\delta D} = h_{ext}$ on $\partial\Omega$:

$$\begin{aligned} \sum_{i \in \mathcal{J}_2} I_{1i} &= \int_{\cup_{i \in \mathcal{J}_2} \partial V_i} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{w} \nabla w \cdot \tau \, ds \\ &= 4 \int_{\cup_{i \in \mathcal{J}_2} \partial U_i} (h_{\delta D} - h_{ext}) \operatorname{Im} \bar{v} \nabla v \cdot \tau \, ds + \int_{\cup_{i \in \mathcal{J}_2} (V_i \setminus U_i)} \nabla^\perp h_{\delta D} \cdot \operatorname{Im} \bar{w} \nabla w \, dx \\ &= 4 \int_{\cup_{i \in \mathcal{J}_2} U_i} \nabla^\perp (h_{\delta D} - h_{ext}) \cdot \operatorname{Im} \bar{v} \nabla v \, dx + 4 \int_{\cup_{i \in \mathcal{J}_2} U_i} (h_{\delta D} - h_{ext}) \operatorname{Im}(\nabla^\perp \bar{v} \cdot \nabla v) \, dx + o(1) \\ &= o(1). \end{aligned} \quad (115)$$

The external magnetic field here plays the same role as $h_{\delta D}(b^i)$ in (114), that is:

$$|h_{\delta D}(x) - h_{ext}| \leq \|\nabla h_{\delta D}\|_{L^\infty(\Omega)} \cdot 2r_i \leq O(\delta) \quad (116)$$

in B_i for $B_i \cap \partial\Omega \neq \emptyset$ because $h_{\delta D} = h_{ext}$ on $\partial\Omega$.

Combining the estimates we obtain

$$\sum_{i \in \mathcal{J}} I_{1i} = \sum_{i \in \mathcal{J}_1} 2\pi d_i (h_{\delta D}(b^i) - h_{ext}) + o(1), \quad (117)$$

thus concluding the proof. \square

Putting together (88), (94), and (95) we get

$$F_\delta[v, B; G] + \pi d \left(\log \frac{\delta^2}{d\varepsilon} - C \right) \leq 2\pi \sum_{i \in \mathcal{J}_1} (h_{ext} - h_{\delta D}(b^i)) d_i + 2\pi \sum_{j=1}^N D_v^j (h_{ext} - H_R^j) + O(1), \quad (118)$$

where $d = \sum_{i \in \mathcal{J}} |d_i|$ as before. This inequality holds under the assumption that d is nonzero. If, on the other hand, d equals zero, the term $\pi d \left(\log \frac{\delta^2}{d\varepsilon} - C \right)$ should be dropped.

In the following lemma we obtain the lower bound for F_δ that allows us to show that there are no bulk vortices, i.e., $d_i = 0$.

Lemma 5. *There exists a $\delta_0 > 0$ such that, for any $\delta \leq \delta_0$, there are no bulk vortices inside the domain $\Omega \setminus \overline{S}$. Moreover, there exist an $\alpha > 1$ and an $\delta \ll R' \ll 1$ such that the following inequality holds:*

$$\sum_{j=1}^N \left[\pi(1-\theta)^2(|\log \delta| - |\log R'| + O(\delta))(D_v^j)^2 - 2\pi D_v^j(h_{ext} - H_R^j) \right] \leq O(1). \quad (119)$$

Proof. Fix $\alpha > 1$ and consider two cases:

1. $\sum_{j=1}^N |D_v^j| \leq \alpha \sum_{i \in \mathcal{J}} |d_i|$. The leading term in (118) is $\pi d |\log \varepsilon|$ on the left hand side and it cannot be bounded by the right hand side if $d \neq 0$ because the leading term there is of order $d \cdot O(|\log \delta|)$. Therefore $d = 0$, and there are no bulk vortices and all $D_v = 0$.
2. $\sum_{j=1}^N |D_v^j| > \alpha \sum_{i \in \mathcal{J}} |d_i|$. We need an additional lower bound on the energy $F_\delta[v, B; G]$.

To estimate $F_\delta[v, B; G]$, we integrate over circles $\gamma_r^j = \partial B(a^j, r)$ around the holes ω_δ^j with $r > R$. If $|u| \neq 0$ on γ_r^j for some $r > R$, we can define the degree on γ_r^j via

$$D_r^j = \deg(u, \gamma_r^j) = \deg(v, \gamma_r^j) \quad (120)$$

Denote

$$\mathfrak{R} = \{r \in (R, R_{max}) : |u| > 1 - \theta \text{ on } \gamma_r^j \text{ for all } j = 1 \dots N\}, \quad (121)$$

where θ is specified in the Ball Construction Method and R_{max} plays the same role as in Lemma 1, i.e., it is the maximal radius r such that $B(a^j, r)$ are disjoint and do not intersect $\partial\Omega$. The total degree on $\partial\Omega$ is the sum of the degrees of all vortices. Since $D_r^j = D_v^j$ by definition of D_v^j , we have

$$\sum_{j=1}^N |D_r^j| \geq \sum_{j=1}^N |D_v^j| - \sum_{i \in \mathcal{J}} |d_i| \geq \frac{\alpha - 1}{\alpha} \sum_{j=1}^N |D_v^j|. \quad (122)$$

Using the definition of the degree and the Divergence Theorem for $r \in \mathfrak{R}$ we get

$$2\pi D_r^j - \int_{B_r^j} \operatorname{curl} B \, dx = \int_{\gamma_r^j} \nabla \Phi \cdot \tau - B \cdot \tau \, dS = \int_{\gamma_r^j} (\nabla \Phi - B) \cdot \tau \, dS \quad (123)$$

or

$$2\pi D_r^j = \int_{\gamma_r^j} (\nabla \Phi - B) \cdot \tau \, dS + \int_{B_r^j} \operatorname{curl} B \, dx = I_1(r) + I_2(r) \quad (124)$$

for any $j = 1 \dots N$. Here $B_r^j = B(a^j, r)$ and $v = |v|e^{i\Phi}$. The following estimates

$$I_1^2 \leq \operatorname{meas}(\gamma_r^j) \int_{\gamma_r^j} |\nabla \Phi - B|^2 \, dS \leq 2\pi r \int_{\gamma_r^j} \frac{|(\nabla - iB)v|^2}{|v|^2} \, dS \leq \frac{2\pi r}{(1-\theta)^2} \int_{\gamma_r^j} |(\nabla - iB)v|^2 \, dS, \quad (125)$$

$$I_2^2 \leq \operatorname{meas}(B_r^j) \int_{B_r^j} |\operatorname{curl} B|^2 \, dx \leq C_1 |\log \delta|^2 r^2, \quad (126)$$

hold since $|v| > 1 - \theta$ by the Ball Construction Method. Further

$$\begin{aligned} 4\pi^2 (D_r^j)^2 &= (I_1(r) + I_2(r))^2 \\ &\leq \frac{2\pi r}{(1-\theta)^2} \int_{\gamma_r^j} |(\nabla - iB)v|^2 \, dS + 2C_1 |\log \delta|^2 r^2 \cdot I_1 + C_1 |\log \delta|^2 r^2 \end{aligned} \quad (127)$$

for $r \in \mathfrak{R}$. Now, divide both sides of (127) by r and integrate outside of the “bad” disks from R to R'

$$\begin{aligned}
4\pi^2 \int_{(R,R') \cap \mathfrak{R}} \frac{(D_r^j)^2}{r} dr &\leq \frac{2\pi}{(1-\theta)^2} \int_{(R,R') \cap \mathfrak{R}} \int_{\gamma_r^j} |(\nabla - iB)v|^2 dS dr \\
&\quad + 2C_1 |\log \delta|^2 \cdot \int_{(R,R') \cap \mathfrak{R}} I_1 r dr + C_1 |\log \delta|^2 \frac{r^2}{2} \Big|_R^{R'} \\
&\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; B_{R'}^j] + \frac{C_1}{2} |\log \delta|^2 R'^2 \\
&\quad + 2C_1 |\log \delta|^2 \cdot R' \cdot \sqrt{\pi R'^2} \cdot \left(\int_{(R,R') \cap \mathfrak{R}} \int_{\gamma_r} \frac{|(\nabla - iB)v|^2}{|v|^2} dS dr \right)^{1/2} \\
&\leq \frac{4\pi}{(1-\theta)^2} F_\delta[v, B; K^j] + \frac{C_1}{2} |\log \delta|^2 R'^2 + \frac{C_3}{1-\theta} |\log \delta|^3 R'^2, \tag{128}
\end{aligned}$$

since $|v| > 1 - \theta$ by the definition of \mathfrak{R} . Here $R' \ll R_{max}$ that will be prescribed later on and K^j is a union of concentric rings around j th hole:

$$K^j = \bigcup_{r \in (R,R') \cap \mathfrak{R}} \gamma_r^j = \bigcup_{r \in (R,R') \cap \mathfrak{R}} \partial B(a^j, r). \tag{129}$$

Notice that all K^j are disjoint since $R' \ll R_{max}$ and $K^j \subset G$ for all $j = 1 \dots N$.

In order to obtain the lower bound for F_δ we divide both sides in (128) by $4\pi/(1-\theta)^2$:

$$\pi(1-\theta)^2 \int_{(R,R') \cap \mathfrak{R}} \frac{(D_r^j)^2}{r} dr \leq F_\delta[v, B; K^j] + \frac{C_1(1-\theta)^2}{8\pi} |\log \delta|^2 R'^2 + \frac{C_3(1-\theta)}{4\pi} |\log \delta|^3 R'^2. \tag{130}$$

We can choose

$$R' = C\zeta^{1/2} |\log \delta|^{-2} \gg R \tag{131}$$

and an appropriate constant C such that for $\zeta = |\log \delta|^{-1} = o(1)$ the sum of last two terms in (130) is less than ζ for small δ . Notice, that $\text{meas}((R, R') \setminus \mathfrak{R}) < \delta^2$ by the Ball Construction Method and $R \leq \delta + \delta^2$. Therefore

$$\begin{aligned}
\sum_{j=1}^N \int_{(R,R') \cap \mathfrak{R}} \frac{(D_r^j)^2}{r} dr &\geq \frac{(\alpha-1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_v^j|^2 \log r \Big|_{\delta+2\delta^2}^{R'} \\
&\geq \frac{(\alpha-1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_v^j|^2 (|\log \delta| - |\log R'| + O(\delta)). \tag{132}
\end{aligned}$$

Thus we can combine (130) and (132) to express the lower bound for $F_\delta[v, B; G]$ in terms of the additional degrees D_v^j :

$$\begin{aligned}
F_\delta[v, B; G] &\geq \sum_{j=1}^N F_\delta[v, B; K^j] \\
&\geq \pi(1-\theta)^2 \frac{(\alpha-1)^2}{\alpha^2} \frac{1}{N} \sum_{j=1}^N |D_v^j|^2 (|\log \delta| - |\log R'| + O(\delta)) - \zeta. \tag{133}
\end{aligned}$$

Substituting $\zeta = |\log \delta|^{-1}$ and combining (133) with (118), we get

$$\begin{aligned} & \sum_{j=1}^N \left(\frac{1}{N} \pi (1-\theta)^2 \frac{(\alpha-1)^2}{\alpha^2} (|\log \delta| - |\log R'| + O(\delta)) (D_v^j)^2 - 2\pi (h_{ext} - H_R^j) D_v^j \right) \\ & \leq -\pi \sum_{i \in \mathcal{I}_1} |d_i| (|\log \varepsilon| - 2|\log \delta| + |\log d| - C) + 2\pi \sum_{i \in \mathcal{I}_1} (h_{ext} - h_{\delta D}(b^i)) d_i + O(1). \end{aligned} \quad (134)$$

Compare the order of the leading terms in (134):

$$\sum_{j=1}^N \left(A |\log \delta| (D_v^j)^2 - O(|\log \delta|) D_v^j \right) \leq -d |\log \varepsilon| + O(1) \quad (135)$$

with $A > 0$. The left hand side of (135) is a sum of quadratic functions in D_v^j with positive leading coefficients:

$$q_j(D_v^j) = A |\log \delta| (D_v^j)^2 - O(|\log \delta|) D_v^j. \quad (136)$$

The values of parabolas q_j are bounded from below by the values at their vertices

$$t^j = \frac{O(|\log \delta|)}{2A |\log \delta|} = O(1), \quad (137)$$

that are themselves bounded. Therefore

$$-d |\log \varepsilon| + o(1) \geq \sum_{j=1}^N q_j(D_v^j) \geq \sum_{j=1}^N q_j(t_j) = O(|\log \delta|). \quad (138)$$

Since $|\log \varepsilon| \gg |\log \delta|$, the inequality (138) can hold only if $d = 0$, i.e. there are no bulk vortices. This, in turn, implies that $D_r^j = D_v^j$ and the inequality (122) is no longer needed. It simplifies the lower bound (132) and yields the desired inequality. \square

6 Proof of Theorem 1: Equality of Degrees

Proof. To finish the proof of Theorem 1 we need to show that all $D_v^j = 0$. We start with the quadratic inequality for D_v^j obtained in Lemma 5:

$$\sum_{j=1}^N \left[\pi (1-\theta)^2 (|\log \delta| - |\log R'| + O(\delta)) (D_v^j)^2 - 2\pi D_v^j (h_{ext} - H_R^j) \right] \leq O(1), \quad (139)$$

where $H_R^j = h_{ext} \xi_0(a^j) + D^j K_0(R)$. This inequality has the same structure as the quadratic functional in S^1 -valued case: there are no mixed terms $D_v^i D_v^j$. Therefore we can find zeros for each $j = 1 \dots N$ separately.

Fix $1 \leq j \leq N$. Clearly, $D_v^j = 0$ is one of two roots of

$$\pi (1-\theta)^2 (|\log \delta| - |\log R'| + O(\delta)) (D_v^j)^2 - 2\pi D_v^j (h_{ext} - H_R^j) = 0. \quad (140)$$

Since $K_0(R) = |\log \delta| + O(1)$ and

$$D^j = \llbracket \sigma (1 - \xi_0(a^j)) \rrbracket, \quad (141)$$

we can calculate the coefficient for the linear term in (139):

$$\begin{aligned} -2\pi(h_{ext} - H_R^j) &= -2\pi(\sigma |\log \delta| - \sigma |\log \delta| \xi_0(a^j) - \llbracket \sigma (1 - \xi_0(a^j)) \rrbracket |\log \delta|) + O(1) \\ &= -2\pi |\log \delta| (\sigma (1 - \xi_0(a^j)) - \llbracket \sigma (1 - \xi_0(a^j)) \rrbracket) + O(1). \end{aligned} \quad (142)$$

Since $\llbracket \cdot \rrbracket$ is the nearest integer, we have

$$\left| \sigma (1 - \xi_0(a^j)) - \llbracket \sigma (1 - \xi_0(a^j)) \rrbracket \right| \leq \frac{1}{2} - \xi, \quad (143)$$

assuming the uniqueness condition (10) and taking

$$\xi = \min_{j=1 \dots N} \text{dist} \left(\sigma (1 - \xi_0(a^j)), \mathbb{Z} + \frac{1}{2} \right) > 0. \quad (144)$$

The second zero of (140) can be estimated as follows:

$$|t_j| = \left| \frac{-2\pi(\sigma (1 - \xi_0(a^j)) - \llbracket \sigma (1 - \xi_0(a^j)) \rrbracket) + o(1)}{\pi(1 - \theta)^2 + o(1)} \right| < \frac{1 - 2\xi}{(1 - \theta)^2 + o(1)} + o(1). \quad (145)$$

Having ξ fixed and $\delta < \delta_0$ sufficiently small, we can always take $\theta > 0$ small enough to make sure that $|t_j| < 1 - \xi$.

Since D_v^j can take only integer values, if at least one D_v^j is nonzero, the left hand side of (139) becomes strictly positive of order $O(\log \delta)$. This contradiction finishes the proof of main theorem yielding

$$D_v^j = 0 \text{ or } D_{\delta, \varepsilon}^j = D^j \quad (146)$$

for all $j = 1 \dots N$. □

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A Appendix. Gradient estimate

Lemma 6. *Let u solve the Poisson equation with Dirichlet boundary conditions in $\Omega_\delta = \Omega \setminus \cup_{j=1}^N \omega_\delta^j$ with $\omega_\delta^j = B(a^j, \delta)$, that is*

$$\begin{cases} -\Delta u = f & \text{in } \Omega_\delta, \\ u = g & \text{on } \partial\Omega, \\ u = g_j & \text{on } \partial\omega_\delta^j, \end{cases} \quad (147)$$

where g and g_j are smooth functions that are defined in the whole of Ω_δ . Then

$$\|\nabla u\|_{L^\infty(\Omega_\delta)} \leq C \left(\frac{1}{\delta} \|u\|_{L^\infty(\Omega_\delta)} + \|f\|_{L^\infty(\Omega_\delta)} + \|\Delta g\|_{L^\infty(\Omega)} + \delta \sum_{j=1}^N \|\Delta g_j\|_{L^\infty(\Omega_\delta)} \right). \quad (148)$$

Proof. The proof is based on lemmas A.1 and A.2 from [8]. Consider the three cases: the point $x_0 \in \Omega_\delta$ is far from the boundaries of $\partial\Omega_\delta$, it is close to $\partial\Omega$, and it is close to $\partial\omega_\delta^j$ for some $j = 1 \dots N$. The first case when $x_0 \in K \subset\subset \Omega_\delta$ is resolved in Lemma A.1 [8] and the second case, when x_0 is close to $\partial\Omega$, can be deduced from Lemma A.2 using $\tilde{u} = u - g$. The results of both lemmas can be merged together in the following estimate:

$$|\nabla u(x_0)| \leq C (\|u\|_{L^\infty} + \|f\|_{L^\infty} + \|\Delta g\|_{L^\infty}) \quad \text{a.e.} \quad (149)$$

when $\text{dist}(x_0, \partial\omega_\delta^j) > m > 0$ with some fixed m independent of δ .

The third case is specific to our setting. Let x_0 be close to one of the holes: $\text{dist}(x_0, \partial\omega_\delta^j) \leq m$ for some $j = 1 \dots N$. Without loss of generality assume $a^j = 0$. We introduce the new spatial variable $y = \frac{x}{\delta}$ to rescale the domain so that the ω_δ^j becomes $B(0, 1)$ and x_0 becomes y_0 . The Poisson equation in new coordinates becomes

$$-\Delta_y u = \delta^2 f. \quad (150)$$

If $\text{dist}(y_0, \partial B(0, 1)) > m$, we apply Lemma A.1 from [8] again. It gives us the estimate for $|\nabla_y u(y_0)|$:

$$|\nabla_y u(y_0)| \leq C (\|u\|_{L^\infty} + \delta^2 \|f\|_{L^\infty}) \quad (151)$$

that in turn implies the estimate for $|\nabla_x u(x_0)|$:

$$|\nabla_x u(x_0)| = \frac{1}{\delta} |\nabla_y u(y_0)| \leq \frac{C}{\delta} \|u\|_{L^\infty(\Omega_\delta)} + C\delta \|f\|_{L^\infty(\Omega_\delta)}. \quad (152)$$

Finally, we apply Lemma A.2 to $\tilde{u}_j = u - g_j$ that satisfies the problem

$$\begin{cases} -\Delta_y \tilde{u}_j = \delta^2 f + \Delta_y g_j & \text{in } B(0, 1+m) \setminus \overline{B(0, 1)}, \\ \tilde{u}_j = h_j & \text{on } \partial B(0, 2+m), \\ \tilde{u}_j = 0 & \text{on } \partial B(0, 1). \end{cases} \quad (153)$$

where $h_j(y) = u(y) - g_j(y)$. Since the proof of Lemma A.2 uses only local estimates and y_0 is far from the $\partial B(0, 2+m)$, the function h_j does not play a role for the estimate of $|\nabla_y u(y_0)|$. It yields the estimate

$$|\nabla_y u(y_0)| \leq C (\|u\|_{L^\infty} + \delta^2 \|f\|_{L^\infty} + \|\Delta_y g_j\|_{L^\infty}). \quad (154)$$

Going back to x we obtain

$$|\nabla_x u(x_0)| \leq \frac{C}{\delta} \|u\|_{L^\infty} + C\delta (\|f\|_{L^\infty} + \|\Delta_x g_j\|_{L^\infty}). \quad (155)$$

Merging all the estimates we finish the proof. \square